International Journal of Theoretical Physics, Vol. 12, No. 3 (1975), pp. 169–176

Zero Value of the Schwarzschild Mass for an Asymptotically Euclidian System of Gravitational Waves

V. N. FOLOMESHKIN

Institute of High Energy Physics, Serpukhov, Moscow District, U.S.S.R.

Received: 3 *June* 1974

Abstract

A class of asymptotically Euclidian space-times is shown to exist for which the Schwarzschild mass is equal to zero. The coordinate atlases of these space-times satisfy two additional conditions: $\partial_\kappa(-gg^{0k}) = 0$ and $\Gamma^0_{ik}\partial_0 g^{ik} - \Gamma^k_{ik}\partial_0 g^{0i} = 0$. In a T-orthogonal metric $ds^2 = g_{00}dt^2 - g_{\alpha\beta}dx^\alpha dx^\beta$ these conditions take a simple form: $\partial_0(\det g_{\alpha\beta}) = 0$ and $(\partial_0 g^{\alpha\beta})(\partial_0 g_{\alpha\beta}) = 0$.

1. *Introduction*

In most papers on general relativity in which the problem of energy momentum was considered (Lorentz, 1915, 1916; Gilbert, 1915, 1924; Einstein, 1916; Klein, 1917, 1918; Weyl, 1917, 1918; and additional references in the papers of Bessel-Hagen (1921), Eddington (1923), Tolman (1930, 1934), Born (1937), Pauli (1941), Iskraut (1942), de Wet (1947), Schrödinger (1948, 1951), Bergmann *et al.* (1948-1956), Hill (1951), Goldberg (1953, (1958), Trautman (1956, 1964), Møller (1958, 1959, 1961), Mitskevich (1958) and Fletcher (1960)) the 'energy' was *defined* with the generalised 'energy-momentum' complex ('pseudo-tensor'). It is known that integral quantities obtained with the generalised complex have no definite transformation properties. The generalised complex is a generalisation, for the case of a continuous medium of the notion of generalised momentum which, in the usual case, has no relation to the usual energy-momentum vector. The physical sense of the generalised complex depends on the choice of a particular coordinate atlas. Therefore the appearance of the Lorentz (1916c, d) paradox, considered also by Schrödinger (1918), Bauer (1918) and Møller (1964), is not astonishing. It is not amusing when some authors obtain negative values for the gravitational wave 'energy' (Hu, 1947; Peres, 1959; Havas & Goldberg, 1962; Sexl, 1966; Petrov, Piragas & Dobrovolsky, 1968), and others obtain a zero value (Brdicka, 1951; Infeld, 1953, 1956, 1959; Sheidegger, 1951, 1953, 1955; Møller, 1958b; Trautman, 1958a and b; Pirani, 1961; Misner, 1963; Cahen & Sengier-Diels, 1963; Langer, 1963; Kuchar & Langer, 1963; Shirokov, 1972; Folomeshkin, 1970), and many others obtain a positive value.

Folomeshkin (1967, 1969, 1971, 1972), Ray (1968) and Plybon (1971) considered the covariant formulation of the differential conservation laws in Riemannian space. (As Pauli (1921) noted, 'in the general case and in the principal problems of a theory general covariance is necessary'). It follows from this formulation that the canonical energy-momentum tensor of gravitational waves is equal to zero (Folomeshkin, 1967, 1969, 1971). Therefore it seems possible that in some cases the Schwarzschild mass (S-mass) of an asymptotically Euclidian system of gravitational waves can be equal to zero.

Brill (1959), Araki (1959), Arnowitt, Deser & Misner (1960), Brill, Deser & Faddeev (1968) and Brill & Deser (1968) have shown that the S-mass of asymptotically Euclidian space-time without source ($T_i^k = 0$) is positive ($m \geqslant 0$). The weak features of the derivations of these results (the assumption that the usual extremum theorems for functions are also valid for functionals, the assumption that there exist non-trivial asymptotically Euclidian solutions which are non-singular, use of the weak field approximation, use of a particular coordinate atlas) were noted by the authors themselves.

In the preceding paper (Folomeshkin, 1974) I have shown that there exists a class of asymptotically Euclidian space-times for which $m = 0$ in the t-symmetrical case, i.e. when $\partial_0 g_{ik} = 0$. The coordinate atlases of this class of space-times satisfy a simple additional condition (in the t-orthogonal metric) $\partial_0(\det g_{\alpha\beta}) = 0$. The coordinate atlas of Weyl–Levy-Civita which was used, for example, by Brill (1959) does not satisfy this condition.

The results of Brill (1959) and Folomeshkin (1974) show that the choice of some additional 'coordinate' conditions determines not only the coordinate atlas but the space-time also, i.e. some class of the solutions of the Einstein equations which transform one into another under the isometries of the connected four-dimensional para-compact Hausdorff pseudo-Reimannian manifold with a normal signature (+−−−).

In the present paper I will show that a more general class of the asymptotically Euclidian space-times exists, for which the S-mass is equal to zero when $T_i^k = 0$.

It is necessary to underline that this result does not prohibit the existence of the different non-isometric (but asymptotically Euclidian also) space-times with a non-zero value of the S-mass when $T_i^k = 0$. And this result does not mean that: (1) the space-time is flat; (2) the gravitational waves are non-detectable; (3) the energy of the source (detector) of the gravitational waves is not changed in the process of the emission (absorption) of the waves. The mechanical effect of the gravitational waves is determined by the curvature tensor and has no direct relation to the S-mass (energy) of the gravitational waves.

2. *Definition of the S-Mass*

The scalar curvature can be divided into two parts, $R = G + \partial_k \omega^k / \sqrt{(-g)}$, where $G = g^{ik}(\Gamma^n_{im}\Gamma^m_{kn} - \Gamma^m_{nm}\Gamma^n_{ik})$ and $\omega^k = \sqrt{(-g)}(g^{mn}\Gamma^k_{mn} - g^{kn}\Gamma^m_{nm})$. In the arbitrary coordinate system the equations $8\pi T^k_i = R^k_i - \delta^k_i R/2$ can be written in the form

$$8\pi\sqrt{(-g)}(T^k_i - \delta^k_i T/2) = -\partial_n L^{kn}_i + \left(\partial_i \omega^k + \frac{\partial\sqrt{(-g)}G}{\partial g^{mn}_k} g^{mn}_i\right) \Big/ 2$$

where $g^{mn}_i = \partial_i g^{mn}$ and $L^{kn}_i = \sqrt{(-g)}(g^{mn}\Gamma^k_{im} - g^{km}\Gamma^n_{im})/2$. In the static case we have

$$8\pi\sqrt{(-g)}(T^0_0 - T/2) = -\partial_\alpha L^{0\alpha}_0, \qquad (\alpha = 1, 2, 3) \tag{2.1}$$

Assuming that as $r \to \infty$ we have an asymptotically Euclidian metric, e.g. in the standard form

$$ds^2 = (1 - 2m/r)\,dt^2 - dr^2/(1 - 2m/r) - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{2.2}$$

we obtain that

$$-\frac{1}{4\pi}\int \partial_\alpha L^{0\alpha}_0\, dV = m \tag{2.3}$$

and therefore

$$m = 2\int (T^0_0 - T/2)\sqrt{(-g)}\, dV \tag{2.4}$$

This is a well-known result (Nordström, 1918; Tolman, 1930; Whittaker, 1935; Ruse, 1935; Klark, 1947 and Papapetrou, 1947). Equations (2.3) and (2.4) are the generalisation to the non-Euclidian space-time of the usual Gauss theorem (Whittaker, 1935).

We will consider equations (2.2)–(2.4) as the definition of the S-mass of an asymptotically Euclidian system.

For an arbitrary non-static system we have, instead of equation (2.1),

$$-\partial_\alpha L^{0\alpha}_0 = 8\pi\sqrt{(-g)}(T^0_0 - T/2) - \left(\partial_0\omega^0 + \frac{\partial\sqrt{(-g)}G}{\partial g^{mn}} g^{mn}_0\right) \Big/ 2$$

and instead of equation (2.4)

$$m = 2\int (T^0_0 - T/2)\sqrt{(-g)}\, dV - \frac{1}{8\pi}\int\left(\partial_0\omega^0 + \frac{\partial\sqrt{(-g)}G}{\partial g^{mn}_0} g^{mn}_0\right) dV \tag{2.5}$$

3. *S-Mass and Generalised 'Energy-momentum' Complex*

Equation (2.5) is simply the integral form for one of the Einstein equations in the asymptotically Euclidian case. It is interesting to compare equality (2.5) with the definition of mass with a generalised complex t_i^k (for example, with the Einstein complex)

$$\mu = \int (T_0^0 + t_0^0)\sqrt{(-g)}\, dV \tag{3.1}$$

$$t_i^k = \left(G\delta_i^k - \frac{G}{g_k^{mn}} g_i^{mn} \right) \Big/ (16\pi)$$

Let us substitute in the right-hand side of equation (2.5), instead of $(T_0^0 - T)$, its expression according to the field equations. We obtain

$$m = \int (T_0^0 + t_0^0)\sqrt{(-g)}\, dV - \frac{1}{8\pi}\int \partial_\alpha (L_0^{0\alpha} - \omega^\alpha/2)\, dV$$

We see that the quantity μ defined with equation (3.1) where t_i^k is the Einstein complex is not equal to the S-mass of the system and differs from it by the quantity

$$\Delta = m - \mu = -\frac{1}{8\pi}\int \partial_\alpha (L_0^{0\alpha} - \omega^\alpha/2)\, dV$$

If we calculate Δ in different asymptotically Euclidian coordinate systems we will find that the quantity Δ will depend on the coordinate atlas and can have any value. This known 'paradox' means that the quantity μ, defined according to equation (3.1), has no relation to the S-mass of the system in the general case.

4. *S-Mass of an Asymptotically Euclidian System of Gravitational Waves*

The Einstein equations are known to be invariant under arbitrary isometric diffeomorphisms of space-time. Therefore, in the general case, we can impose some additional conditions on the components of the metric tensor. One of the known subsidiary conditions is the set of harmonic conditions (De Donder, 1921; Lanczos, 1923; Fok, 1961) $\partial_i(\sqrt{(-g)}g^{ik}) = 0$. Other possible subsidiary conditions of this type are as follows:

$$\partial_i(-gg^{ik}) = 0 \tag{4.1}$$

The S-mass of the t-symmetrical asymptotically Euclidian space-time with $T_i^k = 0$, the coordinate atlas of which satisfies conditions (4.1), is equal to zero (Folomeshkin, 1974). This result is valid also for a more broad class of space-times. The coordinate atlases of these space-times satisfy one subsidiary condition only

$$\partial_k(-gg^{0k}) = 0 \tag{4.2}$$

When considering a more general case of the asymptotically Euclidian space-time without source, it is naturally to choose subsidiary conditions which for the t-symmetrical moment of time come to condition (4.2) or (4.1). We choose two subsidiary conditions: condition (4.2) and condition

$$\Gamma^0_{ik}\partial_0 g^{ik} - \Gamma^k_{ik}\partial_0 g^{0i} = 0 \tag{4.3}$$

In the t-orthogonal metric

$$ds^2 = g_{00}\, dt^2 - g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

conditions (4.2) and (4.3) take simple forms:

$$\partial_0(\det g_{\alpha\beta}) = 0 \qquad \text{and} \qquad (\partial_0 g^{\alpha\beta})(\partial_0 g_{\alpha\beta}) = 0$$

Any asymptotically Euclidian coordinate atlas satisfies condition (4.3) in the limit $r \to \infty$ and can easily be transformed to the form which satisfies condition (4.2), i.e. these conditions are consistent with the asymptotically Euclidian character of the space-time.

For the class of coordinate atlases which satisfy conditions (4.2) and (4.3) equation (2.5) takes the simple form (2.4) and we obtain

$$m = 0 \qquad \text{when} \qquad T^k_i = 0 \tag{4.4}$$

Condition (4.3) may be considered as some form of generalisation of the time-symmetry initial condition $\partial_0 g^{ik} = 0$.

5. *Conclusion*

We have shown that there exist a class of asymptotically Euclidian space-times for which $m = 0$ when $T^k_i = 0$. This result does not prohibit the existence of other non-isometric space-times with a non-zero value of the S-mass when $T^k_i = 0$. A specific example of such non-isometric space-time is presented by Brill (1959).

Another simple example, when the solutions of the Einstein equations corresponding to different 'coordinate' conditions correspond, in reality, to different non-isometric space-times, is the known solutions for the static gravitational field of a point particle in the vacuum: the standard Schwarzschild solution (S), the harmonic solution (H), the isotropic solution (I), the axially symmetric Weyl solution. The S-metric is the isometric extension of the H-metric, and the region $r < m$ of the S-metric is placed beyond the boundaries of the space-time with H-atlas (it corresponds to negative values of the radius in the H-metric). The I-space $R^I_+(r) = \{x \in R^I_3 \,|\, r \geqslant 0\}$ is mapped (twice) in the proper subset $R^S_{2m}(r) = \{x \in R^S_3 \,|\, r \geqslant 2m\}$ of the S-space and in the proper subset R^H_m of the H-space. The regions $r < 2m$ in the S-space and $r < m$ in the H-space lies beyond the boundaries of the I-space. These space-times are therefore different non-isometric space-times.

This simple example shows that in the general case the additional conditions are not 'coordinate conditions' but space-time conditions.

As we have already noted, the equality to zero of the S-mass of the space-time without source does not mean that the energy of the source (receiver) is not changed in the process of emission (absorption) of the gravitational waves. Many authors (Bondi, 1957; Bondi, Pirani & Robinson, 1959; Bonnor, 1959; Bondi, van der Burg & Metzner, 1962; Newman & Unti, 1962; Sachs, 1962; Bonnor & Rotenberg, 1961, 1966; Tamburino & Winicour, 1966; Rotenberg, 1972) have shown, without use of the notions 'energy-momentum pseudo-tensor' or 'energy' of the gravitational waves, that the S-mass of the source (receiver) is changed in the process of the emission (absorption) of the gravitational waves.

Acknowledgement

I thank B. A. Arbuzov, D. R. Brill, S. Deser, L. D. Faddeev and S. S. Gershtein for the discussion of the problems considered in the present paper.

References

Araki, H. (1959). *Annals of Physics*, 7, 456.
Arnowitt, S., Deser, S. and Misner, C. (1960). *Annals of Physics*, **11**, 116.
Bauer, H. (1918). *Physikalische Zeitschrift*, **19**, 163.
Bergmann, P. G. and Brunnings, J. H. M. (1948). *Review of Modern Physics*, **21**, 480.
Bergmann, P. G. (1949). *Physical Review*, **75**, 680.
Bergmann, P. G., Penfield, R., Schiller, R. and Zatzkis, H. (1950). *Physical Review*, **80**, 81.
Bergmann, P. G. and Thomson, R. (1953). *Physical Review*, **89**, 400.
Bergmann, P. G. (1956). In, *Jubilee of Relativity Theory*. Birkhauser Verlag, Basel.
Bessel-Hagen, E. (1921). *Mathematische Annalen*, **84**, 258.
Bondi, H. (1957). *Nature (London)*, **179**, 1072.
Bondi, H., Pirani, F. and Robinson, I. (1959). *Proceedings of the Royal Society (London)*, **A251**, 519.
Bondi, H., Van der Burg, M. and Metzner, A. (1962). *Proceedings of the Royal Society (London)*, **269**, 21.
Bonnor, W. B. (1959). *Philosophical Transactions of the Royal Society (London)*, **251**, 233.
Bonnor, W. B. and Rotenberg, M. A. (1961). *Proceedings of the Royal Society (London)*, **A265**, 109.
Bonnor, W. B. and Rotenberg, M. A. (1966). *Proceedings of the Royal Society (London)*, **289**, 247.
Born, M. (1937). *Annales de l'Institut Henri Poincaré*, **7**, 155.
Brdicka, M. (1951). *Proceedings of the Royal Irish Academy*, **A54**, 137.
Brill, D. R. (1959). *Annals of Physics*, **7**, 466.
Brill, D. R., Deser, S. and Faddeev, L. D. (1968). *Physics Letters*, **A-26**, 538.
Brill, D. R. and Deser, S. (1968). *Physical Review Letters*, **20**, 75.
Cahen, M. and Sengier-Diels, J. (1963). *Bulletin de l'Académie r. de Belgique, Classe des Sciences*, **49**, 43.
Capella, M. A. (1961). *Compte Rendu Hebdomadaire des Séances de l'Académie des Sciences, Paris*, **252**, 3940.
De Donder, T. (1921). *La gravifique Einsteinienne*. Paris.
De Wet, J. S. (1947). *Proceedings of the Cambridge Philosophical Society*, **43**, 511.
Eddington, A. S. (1923). *The Mathematical Theory of Relativity*. The University Press, Cambridge.
Einstein, A. (1916). *Sitzungsberichte der Preussischen Academie der Wissenschaften zu Berlin*, **2**, 1111.
Fok, V. A. (1961). *Theory of Space, Time and Gravitation*. Publishing House of Phys.-Math. Literature, Moscow.

Folomeshkin, V. N. (1967). *Yadernaya Fizika*, **6**, 1080.
Folomeshkin, V. N. (1969). *Acta Physica Polonica*, **36**, 439.
Folomeshkin, V. N. (1970). *Acta Physica Polonica*, **B1**, 13.
Folomeshkin, V. N. (1971). Institute of High Energy Physics Preprint, IHEP 71-120, Serpukhov.
Folomeshkin, V. N. (1972). Institute of High Energy Physics Preprint, IHEP 72-92, Serpukhov.
Folomeshkin, V. N. (1974). *International Journal of Theoretical Physics*, Vol. 10, No. 3, p. 145.
Fletcher, J. (1960). *Review of Modern Physics*, **32**, 65.
Goldberg, J. N. (1953). *Physical Review*, **89**, 263.
Goldberg, J. N. (1958). *Physical Review*, **111**, 315.
Havas, P. and Goldberg, J. N. (1962). *Physical Review*, **128**, 398.
Hilbert, D. (1915). *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Math.-Phys. Klasse*, **3**, 395.
Hilbert, D. (1924). *Mathematische Annalen*, **92**, 1.
Hill, E. L. (1951). *Review of Modern Physics*, **23**, 253.
Hu, N. (1947). *Proceedings of the Royal Irish Academy*, **A51**, 87.
Infeld, L. and Sheidegger, A. E. (1951). *Canadian Journal of Mathematics*, **3**, 195.
Infeld, L. (1953). *Canadian Journal of Mathematics*, **5**, 17.
Infeld, L. (1956). *Mathematische Annalen*, **17**, 545.
Infeld, L. (1959). *Annals of Physics*, **16**, 341.
Infeld, L. and Plebansky, J. (1960). *Motion and Relativity*. Pergamon Press, New York.
Iskraut, R. (1942). *Zeitschrift für Physik*, **119**, 659.
Klark, G. L. (1947). *Proceedings of the Royal Society of Edinburgh*, **62**, 412.
Klein, F. (1917). *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Math-Phys. Klasse*, **4**, 469.
Klein, F. (1918). *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Math.-Phys. Klasse*, **2**, 171.
Klein, F. (1918). *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Math.-Phys. Klasse*, **3**, 394.
Komar, A. (1963). *Physical Review*, **129**, 1873.
Kuchar, K. and Langer, J. (1963). *Czechoslovak Journal of Physics*, **B13**, 233.
Lanczos, K. (1923). *Physikalische Zeitschrift*, **23**, 537.
Langer, J. (1963). *Czechoslovak Journal of Physics*, **B13**, 789.
Lopez, C. A. (1965). *Nuovo Cimento*, **A40**, 19.
Lorentz, H. A. (1915). *Verslagen Koninklijke Academie van Wetenschappen te Amsterdam*, **23**, 1073.
Lorentz, H. A. (1916a). *Verslagen Koninklijke Academie van Wetenschappen te Amsterdam*, **24**, 1389.
Lorentz, H. A. (1916b). *Verslagen Koninklijke Academie van Wetenschappen te Amsterdam*, **24**, 1759.
Lorentz, H. A. (1916c). *Verslagen Koninklijke Academie van Wetenschappen te Amsterdam*, **25**, 468.
Lorentz, H. A. (1916d). *Verslagen Koninklijke Academie van Wetenschappen te Amsterdam*, **25**, 1380.
Misner, C. W. (1963). *Physical Review*, **130**, 1590.
Mitskevich, N. V. (1958). *Annalen der Physik*, **1**, 319.
Mϕller, C. (1958a). *Annals of Physics*, **4**, 347.
Mϕller, C. (1958b). In, *Max-Planck Festschrift*, p. 139. Berlin.
Mϕller, C. (1959). *Kongelige Danske Videnskabernes Selskabs, Matematisk-fysiske (Meddelelser)*, **31**, 14.
Mϕller, C. (1961). *Annals of Physics*, **12**, 118, 1961.
Mϕller, C. (1964). In, *Attidel convegno sulla relativita generale: problemi dell energie ondre gravitazionalli*, Vol. 2. Roma.
Morgan, T. A. and Joseph, D. W. (1965). *Nuovo Cimento*, **39**, 494.

Newman, E. T. and Unti, T. W. J. (1962). *Journal of Mathematical Physics,* **3**, 891.
Nordstrom, G. (1918). *Proceedings K. Nederlandse Academie van Wetenschappen, Amsterdam,* **20**, 1076.
Papapetrou, A. (1947). *Proceedings of the Royal Irish Academy,* **A51**, 191.
Pauli, W. (1921). *Relativitatstheorie. Encyklopadie der Mathematik Wissenschaften,* **2**, 539.
Pauli, W. (1941). *Reviews of Modern Physics,* **13**, 203.
Peres, A. (1959). *Nuovo Cimento,* **11**, 644.
Petrov, A. Z., Piragas, K. A. and Dobrovolsky, V. A. (1968). In, *Gravitation and Relativity,* N4-5. University Publishing House, Kazan
Pirani, F. A. E. (1961). In, *Gravitation: Introduction to Current Research.* John Wiley and Sons, New York.
Plybon, B. (1971). *Journal of Mathematical Physics,* **12**, 57.
Ray, J. R. (1968). *Nuovo Cimento,* **A56**, 189.
Rosen, N. (1956). In, *Jubilee of Relativity Theory,* p. 170. Birkhauser Verlag, Basel.
Rotenberg, M. A. (1972). *Journal of Physics A (General Physics)*, **5**, 198.
Sachs, R. K. (1962). *Proceedings of the Royal Society (London)*, **270**, 102.
Schrödinger, E. (1918). *Physikalische Zeitschrift,* **19**, 4.
Schrödinger, E. (1948). *Proceedings of the Royal Irish Academy,* **A54**, 1.
Schrödinger, E. (1951). *Proceedings of the Royal Irish Academy,* **A54**, 79.
Sexl, R. U. (1966). *Physics Letters,* **20**, 376.
Sheidegger, A. E. (1951). *Physical Review,* **82**, 833.
Sheidegger, A. E. (1953). *Review of Modern Physics,* **25**, 451.
Sheidegger, A. E. (1955). *Physical Review,* **99**, 1883.
Shirokov, M. F. (1972a). In, *Gravitation.* Publishing House 'Naukova Dumka', Kiev.
Shirokov, M. F. (1972b). Institute of Theoretickal Physics Preprint, ITP-72-131E, Kiev.
Tamburino, L. A. and Winicour, J. H. (1966). *Physical Review,* **150**, 1039.
Tolman, R. C. (1940). *Physical Review,* **35**, 875.
Tolman, R. C. (1934). *Relativity, Thermodynamics and Cosmology.* Clarendon Press, Oxford.
Trautman, A. (1956). *Bulletin de l'Académie Polonaise des Sciences,* **4**, 665.
Trautman, A. (1958a). *Bulletin de l'Académie Polonaise des Sciences,* **6**, 627.
Trautman, A. (1958b). In, *Lectures on General Relativity.* King's College, London.
Trautman, A. (1964). In, *Lectures on General Relativity.* Prentice-Hall, New Jersey.
Weber, J. and Wheeler, J. (1957). *Review of Modern Physics,* **29**, 509.
Weyl, H. (1917). *Annalen der Physik,* **54**, 117.
Weyl, H. (1918). *Raum-Zeit-Materia.* Springer Verlag, Berlin.
Whittaker, E. T. (1935). *Proceedings of the Royal Society (London)*, **149**, 384.
Zatzkis, H. (1951). *Physical Review,* **81**, 1023.